

EULER-LIKE RECURRENCES FOR SMALLEST PARTS FUNCTIONS

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In memory of Basil Gordon

ABSTRACT. We obtain recurrences for smallest parts functions which resemble Euler's recurrence for the ordinary partition function. The proofs involve the holomorphic projection of non-holomorphic modular forms of weight 2.

1. INTRODUCTION

Let $p(n)$ denote the unrestricted partition function. One of the fundamental results in partition theory is Euler's recurrence, which states that for $n > 0$ we have

$$\sum_k (-1)^k p\left(n - \frac{k(3k+1)}{2}\right) = 0. \quad (1.1)$$

The *smallest parts* function $\text{spt}(n)$, which counts the number of smallest parts in the partitions of n , was introduced by Andrews [4]. This and other smallest parts functions have been studied widely in recent years from a number of perspectives (see, e.g. [1, 2, 5, 7, 8, 9, 10] and the many references therein). Many of the beautiful properties of these functions originate from the fact that the associated generating functions are components of mock modular forms of weight $3/2$.

Here we use the technique of holomorphic projection (as described by Sturm [14] and Gross-Zagier [12]) to derive analogues of (1.1) for smallest parts functions. The basic principle (also used recently in [3] and [11]) is that for a non-holomorphic modular form $f = f^+ + f^-$ written as a sum of holomorphic and non-holomorphic parts, we have $\pi_{\text{hol}}(f) = f^+ + \pi_{\text{hol}}(f^-)$. If one can identify the holomorphic modular form $\pi_{\text{hol}}(f)$ and can compute $\pi_{\text{hol}}(f^-)$ explicitly, then a formula for f^+ results. The simplest such analogue involves $\text{spt}(n)$. The associated holomorphic projection has been described (without proof) by Zagier [15, §6]; for completeness we give a brief account here.

Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

denote Dedekind's eta function and let $E_2(z)$ be the quasimodular weight 2 Eisenstein series on $\text{SL}_2(\mathbb{Z})$. Define

$$F(z) := \sum_{n=1}^{\infty} \text{spt}(n) q^{n - \frac{1}{24}} - \frac{1}{12} \cdot \frac{E_2(z)}{\eta(z)} + \frac{\sqrt{3}i}{2\pi} \int_{-\bar{z}}^{i\infty} \frac{\eta(\tau)}{(z + \tau)^{\frac{3}{2}}} d\tau.$$

2010 *Mathematics Subject Classification.* 11F37, 11P84.

Key words and phrases. Smallest parts functions, holomorphic projection.

The first author was supported by a grant from the Simons Foundation (#208525 to Scott Ahlgren).

Let ε be the multiplier on $\mathrm{SL}_2(\mathbb{Z})$ associated to the eta function. It can be shown (see [6] or [1, §3]) that $F(z)$ is a weak harmonic Maass form of weight $3/2$ on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier $\bar{\varepsilon}$, so the function $\eta(z)F(z)$ transforms like a modular form of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$. For positive integers n , define

$$a(n) := - \sum_{\substack{ab=6n \\ 0 < a < b}} \left(\frac{12}{b^2 - a^2} \right) \cdot a.$$

We have

$$\sum_{n=1}^{\infty} a(n)q^n = q + 2q^2 + q^3 + 2q^4 - q^5 + 3q^6 - 2q^7 + 2q^8 + q^9 + q^{10} + \dots$$

Letting $E_2^*(z)$ denote the non-holomorphic Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$, it can be shown that the holomorphic projection of $\eta(z)F(z) + \frac{1}{12}E_2^*(z)$ is equal to 0. By computing this projection directly (using an argument similar to those given below) one can deduce that

$$\prod_{n=1}^{\infty} (1 - q^n) \cdot \sum_{n=1}^{\infty} \mathrm{spt}(n)q^n = \sum_{n=1}^{\infty} a(n)q^n.$$

In other words, we have the following Euler-like recurrence for $\mathrm{spt}(n)$, which is recorded in a slightly different form by Zagier [15] and Andrews-Rhoades-Zwegers [3, Thm. 11.1].

Theorem 1. *For $n > 0$ we have*

$$\sum_k (-1)^k \mathrm{spt} \left(n - \frac{k(3k+1)}{2} \right) = a(n).$$

We will derive similar recurrences for other smallest parts functions. An *overpartition* is a partition in which the first occurrence of each part may be overlined. Let $\overline{\mathrm{spt}}(n)$ denote the number of overpartitions of n and let $\overline{\mathrm{spt}}\overline{1}(n)$ denote the number of odd smallest parts in the overpartitions of n (see [7]). Define a divisor function $s(n)$ by

$$s(n) := \sum_{d|n} \min \left(d, \frac{n}{d} \right),$$

with the convention that $s(n) = 0$ if $n \notin \mathbb{Z}$. Define

$$b(n) := (-1)^{n+1} \begin{cases} 2s(n) & \text{if } n \text{ is odd,} \\ 4s(n/4) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Then we have the following analogue of (1.1) for $\overline{\mathrm{spt}}\overline{1}(n)$.

Theorem 2. *For $n > 0$ we have*

$$\sum_k (-1)^k \overline{\mathrm{spt}}\overline{1}(n - k^2) = b(n).$$

Theorem 2 is equivalent to the identity

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \sum_{m=1}^{\infty} \overline{\mathrm{spt}}\overline{1}(m)q^m = \sum_{n=0}^{\infty} b(n)q^n = 2q + 4q^3 - 4q^4 + 4q^5 + 4q^7 - 8q^8 + \dots$$

Since we have

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \right)^{-1} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \dots,$$

we obtain the following

Corollary 3. *For all $N > 0$ we have*

$$\overline{\text{spt}1}(N) = \sum_{n+m=N} \bar{p}(n)b(m).$$

Following [8], let $\text{m2}(n)$ denote the number of partitions of n without repeated odd parts, and define $\text{M2spt}(n)$ as the restriction of $\text{spt}(n)$ to these partitions whose smallest part is even. Define

$$c(n) := \sigma(n) - \sigma(n/2) - \frac{1}{2}s(2n) + s(n/2),$$

where $\sigma(n)$ denotes the usual sum of divisors function.

Theorem 4. *For $n > 0$ we have*

$$\sum_{k \geq 0} (-1)^{k(k+1)/2} \text{M2spt}\left(n - \frac{k(k+1)}{2}\right) = (-1)^n c(n).$$

We will prove the theorem by establishing the identity

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} \sum_{m=1}^{\infty} (-1)^m \text{M2spt}(m) q^m = \sum_{n=1}^{\infty} c(n) q^n = q^2 + q^3 + 3q^4 + 3q^5 + 4q^6 + \dots$$

Since

$$\left(\sum_{n=0}^{\infty} q^{n(n+1)/2} \right)^{-1} = \sum_{n=0}^{\infty} (-1)^n \text{m2}(n) q^n = 1 - q + q^2 - 2q^3 + 3q^4 - 4q^5 + 5q^6 + \dots,$$

we obtain the following

Corollary 5. *For all $n > 0$ we have*

$$\text{M2spt}(N) = \sum_{n+m=N} (-1)^m \text{m2}(n) c(m).$$

2. PRELIMINARIES

Let $k \in \mathbb{Z}$. For matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{Q})$ and functions f on the upper half plane we define

$$(f|_k \gamma)(z) := \det(\gamma)^{\frac{k}{2}} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

We say that f has weight k for $\Gamma_0(N)$ if $f|_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$. Let E_2 denote the weight 2 quasi-modular Eisenstein series

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Then the functions

$$E_2^*(z) := E_2(z) - \frac{3}{\pi y} \quad \text{and} \quad E(z) := 2E_2(2z) - E_2(z)$$

have weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(2)$, respectively. Letting $W_2 := \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ denote the Fricke involution, we have $E|_2 W_2 = -E$ and

$$(E_2^*|_2 W_2)(z) = 2E_2^*(2z).$$

Define

$$G(z) := \sum_{n \geq 1} \overline{\mathrm{spt}}(n) q^n + \frac{1}{12} \frac{\eta(2z)}{\eta^2(z)} (E_2(z) - 4E_2(2z)) + \frac{1}{2\sqrt{2}\pi i} \int_{-\bar{z}}^{i\infty} \frac{\eta^2(\tau)/\eta(2\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau \quad (2.1)$$

and

$$\begin{aligned} H(z) := & \sum_{n \geq 1} (-1)^n \mathrm{M2spt}(n) q^{n-\frac{1}{8}} \\ & + \frac{1}{24} \frac{\eta(z)}{\eta^2(2z)} (E_2(2z) - E_2(z)) + \frac{1}{2\pi i} \int_{-\bar{z}}^{i\infty} \frac{\eta^2(2\tau)/\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d\tau. \end{aligned} \quad (2.2)$$

By work of Bringmann, Lovejoy, and Osburn [8], these functions are harmonic weak Maass forms of weight $3/2$ (see, for example, [13] for details). In the notation of [8], $G(z) = -\frac{1}{4}\overline{\mathcal{M}}(z)$ and (correcting a sign error) $H(z) = \overline{\mathcal{M}}_2(z/8)$. From the proof of Lemma 6.1 of [8], we have

$$(-i\sqrt{2}z)^{-\frac{3}{2}} G(-1/2z) = -2^{\frac{1}{4}} H(z). \quad (2.3)$$

We use this fact to obtain the following proposition.

Proposition 6. *The functions*

$$g(z) := \frac{\eta^2(z)}{\eta(2z)} G(z) \quad \text{and} \quad h(z) := \frac{\eta^2(2z)}{\eta(z)} H(z)$$

have weight 2 for $\Gamma_0(2)$.

Proof. The group $\Gamma_0(2)/\{\pm I\}$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. By (2.1) and (2.2) we have $g(z+1) = g(z)$ and $h(z+1) = h(z)$. To check the transformation under $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, we write

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = W_2 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_2^{-1}.$$

Using (2.3) and the fact that $\eta(-1/z) = \sqrt{-iz} \eta(z)$, we find that

$$g(z)|_2 W_2 = 2h(z), \quad (2.4)$$

from which

$$g(z)|_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = g(z).$$

The same is true for $h(z)$, and the proposition follows. \square

We introduce the holomorphic projection operator. Let $k \geq 2$ be an even integer. Suppose that $\phi(z)$ has weight k for $\Gamma_0(N)$ and has Fourier expansion

$$\phi(z) = \sum_{m \in \mathbb{Z}} \alpha(m, y) q^m.$$

Define

$$\pi_{\mathrm{hol}}(\phi) := \sum_{m=1}^{\infty} a(m) q^m,$$

where

$$a(m) := \frac{(4\pi m)^{k-1}}{(k-2)!} \int_0^\infty \alpha(m, y) e^{-4\pi m y} y^{k-2} dy \quad (2.5)$$

(provided that this integral converges). The next lemma is Proposition 5.1 of [12] if $k > 2$. When $k = 2$ it follows from the proof of Proposition 6.2, loc. cit. (note that condition (2.6) ensures that the limit and integral at the bottom of page 296 may be interchanged).

Lemma 7. *Suppose that $k \geq 2$. Suppose that $\phi(z)$ has weight k for $\Gamma_0(N)$ and satisfies*

$$(\phi|_k \gamma)(z) \ll y^{-\varepsilon} \quad \text{as } y \rightarrow \infty$$

for some $\varepsilon > 0$ and for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. If $k = 2$, suppose in addition that for some $\varepsilon' > 0$ we have

$$\alpha(m, y) \ll_m y^{-1+\varepsilon'} \quad \text{as } y \rightarrow 0 \quad \text{for all } m > 0. \quad (2.6)$$

Then $\pi_{\mathrm{hol}}(\phi)$ is a weight k cusp form on $\Gamma_0(N)$.

3. PROOF OF THEOREM 2

Write $G = G^+ + G^-$, where

$$G^-(z) = \frac{1}{2\sqrt{2}\pi i} \int_{-\bar{z}}^{i\infty} \frac{\eta^2(\tau)/\eta(2\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$

is the non-holomorphic part. Since $\eta^2(z)/\eta(2z) = 1 + 2 \sum_{n=1}^\infty (-1)^n q^{n^2}$, a computation gives

$$G^-(z) = \frac{1}{2\pi\sqrt{y}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n n \beta(n^2 y) q^{-n^2},$$

where $\beta(y) := \Gamma(-1/2, 4\pi y)$ is the incomplete gamma function. Then

$$g(z) = \frac{\eta^2(z)}{\eta(2z)} G(z) = \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^\infty \overline{\mathrm{spt}} 1(n) q^n + \frac{1}{12} (E_2(z) - 4E_2(2z)) + \sum_{N \in \mathbb{Z}} B(N, y) q^N,$$

where

$$B(N, y) = \frac{(-1)^N}{\sqrt{\pi}} \begin{cases} 2 \sum_{\substack{n^2-m^2=N \\ m>n \geq 1}} m\beta(m^2 y) + \delta_\square(|N|) \sqrt{|N|} \beta(|N| y) & \text{if } N < 0, \\ \frac{1}{2\sqrt{\pi y}} + 2 \sum_{m \geq 1} m\beta(m^2 y) & \text{if } N = 0, \\ 2 \sum_{\substack{n^2-m^2=N \\ n>m \geq 1}} m\beta(m^2 y) + \delta_\square(N) \frac{1}{\sqrt{\pi y}} & \text{if } N > 0. \end{cases} \quad (3.1)$$

Here $\delta_\square(N) = 1$ if N is a square, and 0 otherwise. Since $\beta(y) \sim (4\pi y)^{-3/2} e^{-4\pi y}$ as $y \rightarrow \infty$, we have

$$\sum_{\substack{n^2-m^2=N \\ n, m \geq 1}} m\beta(m^2 y) \ll y^{-\frac{3}{2}} \sum_{\substack{n^2-m^2=N \\ n, m \geq 1}} \frac{1}{m^2} e^{-4\pi m^2 y},$$

where the implied constants here and in the rest of the paragraph are absolute. Since $n^2 - (n-1)^2 = 2n-1$, the equation $n^2 - m^2 = N$ implies that $n, m \leq (|N|+1)/2$. If $N > 0$,

then this sum is $\ll Ny^{-3/2}$. If $N < 0$ then we have $m^2 > -N$ for each term in the sum, from which it follows that the sum is $\ll |N|y^{-3/2}e^{4\pi Ny}$. We conclude that as $y \rightarrow \infty$, we have

$$B(N, y) \ll \begin{cases} |N| y^{-\frac{3}{2}} e^{4\pi Ny} & \text{if } N < 0, \\ y^{-\frac{1}{2}} + Ny^{-\frac{3}{2}} & \text{if } N \geq 0. \end{cases} \quad (3.2)$$

Define

$$\begin{aligned} \hat{g}(z) &:= g(z) + \frac{1}{6}E(z) + \frac{1}{12}E_2^*(z) \\ &= \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^{\infty} \overline{\text{spt1}}(n)q^n - \frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y)q^N. \end{aligned}$$

By (3.2) we have $\hat{g}(z) \ll y^{-1/2}$ as $y \rightarrow \infty$. From (2.4) we obtain

$$\hat{g}|_2 W_2 = 2h(z) + \frac{1}{6}(E_2(z) - E_2(2z)) - \frac{1}{4\pi y}.$$

Therefore $\hat{g}|_2 W_2 \ll y^{-1}$ as $y \rightarrow \infty$ since $h(z)$ decays exponentially at ∞ .

For $N > 0$, we have the bound

$$B(N, y) \ll_N y^{-\frac{1}{2}} \quad \text{as } y \rightarrow 0$$

since $\lim_{y \rightarrow 0} \beta(y) = -2\sqrt{\pi}$. Therefore we may apply Lemma 7 to obtain

$$\pi_{\text{hol}}(\hat{g}) = 0$$

since there are no nontrivial cusp forms of weight 2 on $\Gamma_0(2)$.

We may also compute $\pi_{\text{hol}}(\hat{g})$ using (2.5). Since π_{hol} leaves holomorphic functions unchanged, we have

$$\pi_{\text{hol}}(\hat{g}) = \frac{\eta^2(z)}{\eta(2z)} \sum_{n=1}^{\infty} \overline{\text{spt1}}(n)q^n + \pi_{\text{hol}} \left(-\frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y)q^N \right).$$

By (2.5) we have

$$\pi_{\text{hol}} \left(-\frac{1}{4\pi y} + \sum_{N \in \mathbb{Z}} B(N, y)q^N \right) = \sum_{N=1}^{\infty} \left(4\pi N \int_0^{\infty} B(N, y)e^{-4\pi Ny} dy \right) q^N.$$

By (3.1), the coefficient of q^N above is

$$(-1)^N 8\sqrt{\pi}N \sum_{\substack{n^2 - m^2 = N \\ n, m \geq 1}} m \int_0^{\infty} \beta(m^2 y) e^{-4\pi Ny} dy + \delta_{\square}(N) (-1)^N 4N \int_0^{\infty} y^{-\frac{1}{2}} e^{-4\pi Ny} dy. \quad (3.3)$$

The second integral evaluates to $\frac{1}{2\sqrt{N}}$ and the first is evaluated using the following lemma. The proof is routine (some care is required to justify the change in the order of integration).

Lemma 8. *If $A, B > 0$ then*

$$\int_0^{\infty} \beta(Ay) e^{-4\pi By} dy = \frac{1}{2\sqrt{\pi}B} \left(\sqrt{1 + \frac{B}{A}} - 1 \right). \quad (3.4)$$

Therefore (3.3) becomes

$$\begin{aligned} & (-1)^N 4 \sum_{\substack{n^2-m^2=N \\ n,m \geq 1}} m \left(\sqrt{1 + \frac{N}{m^2}} - 1 \right) + \delta_{\square}(N) (-1)^N 2\sqrt{N} \\ &= (-1)^N 2 \left(2 \sum_{\substack{n^2-m^2=N \\ n,m \geq 1}} (n-m) + \delta_{\square}(N) \sqrt{N} \right). \end{aligned}$$

It remains to show that this evaluates to $-b(N)$. If $N \equiv 2 \pmod{4}$, then the sum is empty and $\delta_{\square}(N) = 0$. If N is odd, then $n-m$ runs over all divisors of N which are less than \sqrt{N} . In this case we have

$$2 \sum_{\substack{n^2-m^2=N \\ n,m \geq 1}} (n-m) + \delta_{\square}(N) \sqrt{N} = \sum_{d|N} \min \left(d, \frac{N}{d} \right).$$

Finally, if $4|N$ then each $n-m$ is even. Letting $r = \frac{n-m}{2}$ and $s = \frac{n+m}{2}$, we find that

$$\sum_{\substack{n^2-m^2=N \\ n,m \geq 1}} (n-m) = \sum_{\substack{rs=N/4 \\ 0 < r < s}} 2r = \sum_{d|\frac{N}{4}} \min \left(d, \frac{N/4}{d} \right) - \delta_{\square}(N) \sqrt{\frac{N}{4}}.$$

□

4. PROOF OF THEOREM 4

We proceed as in the proof of Theorem 2. Write $H = H^+ + H^-$, where

$$H^-(z) = \frac{1}{2\pi i} \int_{-\bar{z}}^{i\infty} \frac{\eta^2(2\tau)/\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d\tau.$$

Since $\eta^2(2z)/\eta(z) = \sum_{\text{odd } n \geq 1} q^{n^2/8}$, we have

$$H^-(z) = \frac{1}{4\sqrt{\pi}} \sum_{\text{odd } n \geq 1} n\beta \left(\frac{n^2 y}{8} \right) q^{-\frac{n^2}{8}}.$$

Define $\hat{h}(z) := h(z) - \frac{1}{24}(E(z) - E_2^*(z))$. Then (2.2) gives

$$\hat{h}(z) = \frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n \text{M2spt}(n) q^{n-\frac{1}{8}} + \frac{1}{24}(E_2(z) - E_2(2z)) - \frac{1}{8\pi y} + \sum_N C(N, y) q^N,$$

where

$$C(N, y) = \frac{1}{4\sqrt{\pi}} \sum_{\substack{n^2-m^2=8N \\ n,m \geq 1 \text{ odd}}} m\beta \left(\frac{m^2 y}{8} \right).$$

By an argument similar to that which gives (3.2), we find that as $y \rightarrow \infty$ we have

$$C(N, y) \ll \begin{cases} |N| y^{-\frac{3}{2}} e^{4\pi N y} & \text{if } N < 0, \\ y^{-\frac{3}{2}} & \text{if } N = 0, \\ N y^{-\frac{3}{2}} & \text{if } N > 0. \end{cases}$$

Thus we have $\hat{h}(z) \ll y^{-1}$ as $y \rightarrow \infty$. We have

$$\hat{h}|_2 W_2 = \frac{1}{2}g + \frac{1}{24}(E(z) + 2E_2(2z)) - \frac{1}{8\pi y}.$$

Therefore $\hat{h}|_2 W_2 \ll y^{-1/2}$ as $y \rightarrow \infty$ since the constant term of $g(z)$ is $-1/4$ and the constant term of $E(z) + 2E_2(2z)$ is 3. For $N > 0$ we have the bound $C(N, y) \ll_N 1$ as $y \rightarrow 0$. Therefore, we may apply Lemma 7 to conclude that $\pi_{\text{hol}}(\hat{h}) = 0$.

Using (2.5), we find that

$$0 = \pi_{\text{hol}}(\hat{h}) = \frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n \text{M2spt}(n) q^{n-\frac{1}{8}} + \frac{1}{24}(E_2(z) - E_2(2z)) + \sum_{N=1}^{\infty} C(N) q^N, \quad (4.1)$$

where

$$C(N) = \sqrt{\pi} N \int_0^{\infty} \sum_{\substack{n^2-m^2=8N \\ n, m \geq 1 \text{ odd}}} m\beta\left(\frac{m^2 y}{8}\right) e^{-4\pi N y} dy.$$

By Lemma 8 we obtain

$$C(N) = \frac{1}{2} \sum_{\substack{n^2-m^2=8N \\ n, m \geq 1 \text{ odd}}} (n - m).$$

Writing $u = \frac{n-m}{2}$ and $v = \frac{n+m}{2}$ gives

$$C(N) = \sum_{\substack{uv=2N \\ u < v \\ u+v \text{ odd}}} u = \frac{1}{2} s(2N) - s(N/2).$$

From (4.1) we conclude that

$$\frac{\eta^2(2z)}{\eta(z)} \sum_{n=1}^{\infty} (-1)^n \text{M2spt}(n) q^{n-\frac{1}{8}} = \sum_{n=1}^{\infty} c(n) q^n.$$

□

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